HYPER VERSION

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Constraints on higher derivative operators in the Matrix theory effective Lagrangian

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ABSTRACT: The consistency of Matrix theory with supergravity requires that in the large N_c limit terms of order v^4 in the $SU(N_c)$ Matrix effective potential are not renormalized beyond one loop in perturbation theory. For $SU(2)$ gauge group, the required non-renormalization theorem was proven recently by Paban, Sethi and Stern. In this paper we consider the constraints supersymmetry imposes on these terms for groups $SU(N_c)$ with $N_c > 2$. Non-renormalization theorems are proven for certain tensor structures, including the structures that appear in the one-loop effective action. However it is expected other tensor structures can in general be present, which may suffer renormalization at three loops and beyond.

KEYWORDS: M(atrix) Theories, M-Theory, Models of Quantum Gravity.

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Contents

1. Introduction

One of the exciting themes to emerge from recent work is the duality between gauge theories and gravitational theories [1, 2]. A prime example is M-theory in discrete light-cone gauge which is thought to be described by the supersymmetric quantum mechanics of low-energy D-particles [3]. Another example is supergravity in d-dimensional anti-de Sitter space which is conjectured to be described by $d-1$-dimensional large N_c superconformal field theory with $SU(N_c)$ gauge group [2].

In the case of M-theory, we are often interested in an effective action expanded in powers of velocities [4, 5, 6, 7, 8, 9, 10, 11]. The effective action computed using the quantum mechanics can be compared straightforwardly with the supergravity results. For the Matrix theory results to be consistent with supergravity, agreement is required in the large N_c limit. This is one of the key assumptions that enters into the statistical derivation of black hole entropy from Matrix theory [12]. Although exact agreement is required only in the large N_c limit, some remarkable results have been obtained for the leading order terms in the expansion for the $SU(2)$ case. In particular, for the $SU(2)$ quantum mechanics, non-renormalization theorems have been proven for the v^4 and v^6 terms [13, 14].

In this paper we consider the constraints supersymmetry imposes on the v^4 terms for the general $SU(N_c)$ case. It is shown certain tensor structures are not renormalized beyond one loop. However, for more general tensor structures, it is argued renormalization should be expected at order v^4 for $N_c > 2$, which may begin to appear at three-loop order and beyond.

2. Matrix review

The Lagrangian of Matrix theory is

$$L = \frac{1}{g^2}\mathrm{Tr}\ ((D_0X^i)^2 + \frac{1}{2}[X_i, X_j]^2 + i\psi_a D_0\psi_a - \psi_a\gamma^i_{ab}[X_i, \psi_b])\,, \tag{2.1}$$

where all the fields are in the adjoint of $SU(N_c)$. The theory is supersymmetric with respect to 16 supercharges.

We will be interested in the effective Lagrangian of this theory at a point where the X's are diagonal, expanded in a power series in velocity. We will use the basis x_A to denote the elements of the Cartan subalgebra, with $A = 1, \cdots, N_c - 1$. The x^A correspond to the $N_c - 1$ relative displacements of N_c D-particles. The effective Lagrangian schematically takes the form

$$L_{eff} = \frac{1}{g^2}\sum_n v^{2n} f_n(r)\,, \tag{2.2}$$

where perturbatively the f_n are

$$f_n(r) = \Big(\frac{1}{r^4}\Big)^{n-1}\sum_l C_{nl}\Big(\frac{g^2}{r^3}\Big)^l, \tag{2.3}$$

and l is the number of loops [4].

The terms of order v^4 will be the focus of this paper. At one loop, the purely bosonic terms take the simple form

$$L_1 = \frac{15}{16}\Big(\sum_A \frac{v_A^4}{x_A^7} + \sum_{A<B}\frac{(v_A - v_B)^4}{(x_A - x_B)^7}\Big). \tag{2.4}$$

This agrees with the formula obtained from linearized supergravity, using the graviton propagator corresponding to zero longitudinal momentum transfer.

3. Constraints from supersymmetry

The object of this paper is to analyze the constraints supersymmetry imposes on the terms of order v^4 in the effective action. The general strategy for the analysis of will follow that of [13], where it was found that supersymmetry ensures (2.4) receives no corrections for the $SU(2)$ case, even at the non-perturbative level. We consider therefore the supersymmetric variation of the eight fermion terms that arise in the supersymmetric completion of the v^4 term. Part of this variation will be a nine fermion term that cannot be canceled by any other source. Demanding this nine fermion term vanish gives strong constraints on the general form of the eight fermion term that appears in the effective action.

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The supersymmetry transformations take the form

$$\begin{aligned}\delta x^i_A &= -i\epsilon\gamma^i\psi_A + \epsilon N^i_{AB}\psi_B\,,\\ \delta\psi_{aA} &= (\gamma^i v^i_A\epsilon)_a + (M_A\epsilon)_a\,,\end{aligned} \tag{3.1}$$

where i labels the vector of $Spin(9)$ and a the 16 component spinor. The effective Lagrangian is expanded in powers of n, which counts the number of time derivatives plus half the number of fermions. N and M encode the higher order corrections to the supersymmetry transformations.

The first step in the calculation is to show that the metric multiplying the v^2 term in the Lagrangian is necessarily flat. N must vanish at leading order ($n = 1$) since no gauge invariant $Spin(9)$ symmetric terms can be constructed. At leading order, M can be order $n = 2$ and we must check this term vanishes. Let us consider the closure of the supersymmetry algebra

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]x^i_A = -2i\epsilon_2\epsilon_1 v^i_A - i\epsilon_{2a}(\gamma^i M_{abA} + M_{baA}\gamma^i)\epsilon_{1b}\,. \tag{3.2}$$

The last term must vanish for the algebra to close. For matrices with $Spin(9)$ spinor indices, this leads to the condition that $M^A = 0$ for all A at this order. This implies the metric is flat, so corrections to M can begin at order $n = 3$ and corrections to N should begin at order $n = 2$.

Next we must consider the general form of the eight fermion terms that can appear in the effective action. They are built out of the general fermion bilinears

$$\psi^A\Gamma\psi^B, \tag{3.3}$$

where $\Gamma \in \{I, \gamma^i, \gamma^{ij}, \gamma^{ijk}, \gamma^{ijkl}\}$. The γ^i represent the $Spin(9)$ Clifford algebra and the other matrices are defined as

$$\begin{aligned}\gamma^{ij} &= \frac{1}{2!}[\gamma^i, \gamma^j]\\ \gamma^{ijk} &= \frac{1}{3!}(\gamma^i\gamma^j\gamma^k - \gamma^j\gamma^i\gamma^k + \cdots)\\ \gamma^{ijkl} &= \frac{1}{4!}(\gamma^i\gamma^j\gamma^k\gamma^l - \gamma^j\gamma^i\gamma^k\gamma^l + \cdots)\,.\end{aligned} \tag{3.4}$$

The Lorentz indices of a general product of four of the fermion bilinears are contracted with a product of between zero and sixteen x^{iA}. Finally the gauge indices are contracted with a tensor function of the x^{iA}, that respects $Spin(9)$ invariance. The sum of terms must respect the residual Weyl invariance of the underlying $SU(N_c)$ gauge theory. If we represent the x_A as $x_A = e_A - e_{A+1}$, the Weyl group acts by permutations on the set of N objects e^A and e^N. For the tensor functions to agree with perturbation theory we know they must fall off in any direction as the x's go to infinity.

The nine fermion term produced in the supersymmetric variation of the eight fermion term $f^{(8)}\psi^8$ is

$$\gamma^n_{ab}\psi_{aA}\frac{\partial}{\partial x^n_A}(f^{(8)}\psi^8) = 0\,. \tag{3.5}$$

This gives us a set of coupled partial differential equations to solve for the general supersymmetry constraints.

As we will see in a moment, it is convenient to derive a weaker set of second order partial differential equations from (3.5). Act on (3.5) with the operator $\gamma^n_{bc}\frac{d}{d\psi_{cM}}\frac{\partial}{\partial x^n_N}$ to give the equation

$$16\frac{\partial^2}{\partial x^n_M \partial x^n_N}(f^{(8)}\psi^8) - \frac{\partial^2}{\partial x^n_L \partial x^n_N}\psi_{aL}(f^{(8)}\frac{d}{d\psi_{aM}}\psi^8) - \\ -\frac{\partial^2}{\partial x^m_L \partial x^n_N}\psi_{aL}\gamma^{mn}_{ac}(f^{(8)}\frac{d}{d\psi_{cM}}\psi^8) = 0\,. \tag{3.6}$$

Let us recall how the solution of these equations works for the $SU(2)$ case [13]. The most general eight fermion term is

$$\psi\gamma^{ij}\psi\psi\gamma^{jk}\psi\psi\gamma^{lm}\psi\psi\gamma^{mn}\psi\Big(g_1(r)\delta_{in}\delta_{kl} + g_2(r)\delta_{kl}x_i x_n + g_3(r)x_i x_k x_l x_n\Big). \tag{3.7}$$

For the $SU(2)$ case, (3.6) simply reduces to the Laplacian. It is easiest to start by considering the action of the Laplacian on the last term in (3.7). By considering the independent tensor structures that arise after acting with the Laplacian, a decoupled equation is obtained for the unknown function g_3. The solution is a power law $g_3 = c/r^{15}$, with c an undetermined constant. The equations for the other functions are coupled to g_3. Solving these equations yields $g_2 = -4c/13r^{13}$ and $g_1 = 2c/143r^{11}$. The other integration constants that arise give solutions with lower inverse powers of r. These terms must vanish for the solution to agree with perturbation theory in the limit $g \to 0$. The only physically consistent solution has scaling appropriate to the one-loop effective action, therefore these terms must not be renormalized.

The key fact that yields the non-renormalization theorem in the $SU(2)$ case is that a unique term, with the maximal number of x's contracted with fermion bilinears, satisfies an equation that decouples from the other unknown functions. The only solution to this equation has a definite scaling with respect to r which corresponds to one-loop behavior. All the terms with lower number of x's are determined in terms of this term, and the physical boundary conditions.

We would like to generalize this statement to $SU(N_c)$ with $N_c > 2$. Unfortunately, the equations (3.6) in general lead to a complicated system of linear second-order partial differential equations for which the general solution is difficult to construct. Before commenting further on the general case let us consider a special set of tensor structures that may be analyzed explicitly.

These correspond to terms that have a nontrivial limit when some of the fermions are constrained to be equal and the rest vanish. To see why we expect a simplification in this case consider first $N_c - 1$ D-particles moving with the same velocity, which may then be Galilean boosted to zero. The positions of the particles remain generic in this limit. A probe D-particle with some finite velocity v now sees a supersymmetric system, so its worldline action is tightly constrained and we can expect a non-renormalization theorem. The analogous statement for the eight fermion term will be that if a term has a nontrivial limit when the ψ^A for $A>1$ are constrained to vanish, the unknown function of the Casimirs multiplying it should be determined.

This statement may be generalized straightforwardly to the case when $N_c - N_0$ D-particles remain at rest and are probed by a collection of N_0 D-particles all moving with the same velocity v. Rather than taking the non-vanishing ψ^A to lie in a regular $SU(2)$ subgroup of $SU(N_c)$ as above, we simply take the single independent non-vanishing ψ to lie in an non-regular $SU(2)$ subgroup of $SU(N_c)$.

Let us consider then the system of equations (3.6) for the case $M=1$, $N=1$ after setting all $\psi^A = 0$ for $A>1$. The result is

$$\frac{\partial^2}{\partial x_n^1 \partial x_n^1}(\psi_1^8 f^{(8)}) = 0\,. \tag{3.8}$$

Again we will first focus on terms with the maximal number (four) of x_j^1's contracted with the fermion bilinears. Once again we obtain a decoupled equation for the unknown function $g_3(x^A)$ of the x's appearing in this term, which takes the same form as for the $SU(2)$ case. Now, however, the solution of the Laplacian can be singular not just at $x^1 = 0$, but at any point where the corresponding D-particles collide. The solutions should fall off in the limit that $x_1 \to \infty$. The general solution then is

$$g_3 = \frac{c_0}{x_1^{15}} + \sum_{A>1} \frac{c_A}{(x_1 - x_A)^{15}}\,, \tag{3.9}$$

where c_0 and c_A are functions of x_B, for $B>1$. The c's can be determined by considering the limit that $x_1 \to x_B$ for some B or $x^1 \to 0$, when $SU(2)$ gauge symmetry is restored. To agree with the $SU(2)$ result [13], the c's must be independent of the x_A. Since the c's are constant, we see these terms have scaling with x such that they can only be generated at one loop. Likewise, one can make a similar argument for terms with fewer x_1's contracted with fermion bilinears. These are determined using the boundary conditions and the solution for g_3 above. This proves all the terms with a nontrivial limit when $N_c - 2$ of the ψ_A vanish are not renormalized beyond one loop. The same argument goes through unchanged when the non-vanishing ψ lies in an non-regular $SU(2)$ subgroup of $SU(N_c)$. Assuming these eight fermion terms uniquely determine the bosonic terms (a result that is expected, but thus far not rigorously proven, even for the $SU(2)$ case) (2.4) contains all the purely bosonic terms with a nontrivial limit when only one independent velocity parameter appears.

However, more generally there can be terms that vanish when all but one of the ψ_A are zero. In this case it does not appear that the equations (3.6) decouple and allow the solutions to be classified directly. Rather than analyze the second order equations (3.6), it is perhaps more efficient to consider the stronger set of first order equations obtained by acting with $\gamma^i_{ac} x_i^M d/d\psi_c^N$. However these equations likewise do not decouple in any obvious way to allow the solutions to be classified explicitly.

4. Discussion

It is possible then that supersymmetry and gauge invariance alone are not sufficient to uniquely fix all the order v^4 terms in the effective action for $N_c > 2$. For finite N_c we do not expect any additional symmetries to be relevant, so assuming the previous statement is valid, renormalization of these terms can be expected. For large N_c additional symmetries may appear (for example eleven-dimensional Lorentz symmetry as discussed in [15]). It is possible these symmetries are sufficient to restrict the form of the solutions so that only terms of the one-loop form can appear.

A heuristic argument why in general renormalization should be expected can be made as follows. Consider a background of $N_c - 1$ D-particles with generic velocities. If another D-particle is used to probe this background, the worldline action for this particle will not be supersymmetric, so no general non-renormalization theorem would be expected to hold. It seems likely even the non-renormalization theorem at order v^4 will fail in this case. Terms in the worldline action that are not renormalized will be mapped by supersymmetry into terms that must vanish. On the other hand, terms that can be renormalized will be mapped by supersymmetry to terms that at best vanish after summing over a number of contributions. This appears to be the structure of the differential equations discussed above. For the general case the equations do not appear to decouple and lead to a unique solution.

No renormalizations of v^4 terms are present at two-loop order [11] for planar diagrams. We have also checked the non-planar diagrams vanish in this case. Such terms may arise at three-loop order and beyond. It should be noted that if such terms appear in perturbation theory they will be accompanied by powers of N_c that are not subleading at large N_c. This will spoil the agreement between perturbative Matrix theory and low-energy supergravity.

Related terms have been considered in [16] at order v^6 and three loops which apparently renormalize the two-loop v^6 answer of [11], for $SU(N_c)$ with $N_c > 4$, and lead to disagreement with the supergravity result. In contrast, for $SU(2)$, the entire set of v^6 terms satisfies a non-renormalization theorem [14]. If these renormalizations appear for v^6 at three loops, one would expect them to contribute to the imaginary part of higher loop v^4 amplitudes via the unitarity relations.

A disagreement between low-energy supergravity and Matrix theory for sufficiently generic amplitudes may simply be a sign that higher-order counter-terms must be added to the Matrix quantum mechanics to regain eleven-dimensional Lorentz invariance. Of course this is the standard state of affairs in the light-front quantization of quantum chromodynamics [17], where counter-terms are fixed by the requirement of Lorentz invariance. The only novelty in Matrix theory may be that these counter-terms are not needed for certain low-order calculations due to the high degree of supersymmetry. If this is the case, it calls into question the use of Matrix theory as a fundamental definition of M-theory.

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